§3.2 Lorentz Invariance

We are now going to show that Lorentz invariance of the Lagrangian density $\mathcal{L}$ implies Lorentz invariance of scattering amplitudes

Consider inf. Lorentz trfs.

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$$

$$\omega_{\mu\nu} = -\omega_{\nu\mu}$$

$\longrightarrow$ set of conserved currents $\mathcal{M}^{\rho\mu\nu}$:

$$\partial_\rho \mathcal{M}^{\rho\mu\nu} = 0$$

$$\mathcal{M}^{\rho\mu\nu} = -\mathcal{M}^{\rho\nu\mu}$$

$\longrightarrow$ time-independent tensors

$$J^{\mu\nu} := \int d^3x \, \mathcal{M}^{0\mu\nu}$$

$$\frac{d}{dt} J^{\mu\nu} = 0$$

$J^{\mu\nu}$ will turn out to be generators of homogeneous Lorentz trfs.

the fields undergo the matrix trf.:

$$\delta\phi^l = \frac{i}{2}\omega^{\mu\nu}(\mathcal{J}_{\mu\nu})^l{}_m \phi^m \quad (1)$$

where $\mathcal{J}_{\mu\nu}$ are a set of anti-sym. matrices satisfying algebra of hom. Lorentz group:

$$[\mathcal{J}_{\mu\nu}, \mathcal{J}_{\rho\sigma}] = i\mathcal{J}_{\rho\nu}\eta_{\mu\sigma} - i\mathcal{J}_{\sigma\nu}\eta_{\mu\rho} - i\mathcal{J}_{\mu\sigma}\eta_{\nu\rho} + i\mathcal{J}_{\mu\rho}\eta_{\nu\sigma}$$

<u>examples</u>:
- scalar field: $\delta\phi=0$, $\mathcal{J}_{\mu\nu}=0$
- vector field: $\delta V_k = \omega_k{}^\lambda V_\lambda$,

  so $(\mathcal{J}_{\rho\sigma})_k{}^\lambda = -i\eta_{\rho k}\delta_\sigma^\lambda + i\eta_{\sigma k}\delta_\rho^\lambda$
- Dirac field:

  $$\mathcal{J}_{\mu\nu} = e^{\frac{i}{4}[\sigma_\mu, \sigma_\nu]}$$

The derivative of ϕ^l transforms as

$$\delta(\partial_k \phi_l) = \frac{1}{2}i\omega^{\mu\nu}(\mathcal{J}_{\mu\nu})_l{}^m \partial_k \phi_m \quad (2)$$
$$+ \omega_k{}^\lambda \partial_\lambda \phi_l$$

vector index

$\mathcal{L}$ is invariant under combined trfs. (1) and (2), so

$$0 = \frac{\delta \mathcal{L}}{\delta \phi^\ell} \frac{i}{2} \omega^{\mu\nu} (\mathcal{J}_{\mu\nu})^\ell{}_m \phi^m \tag{3}$$
$$+ \frac{\delta \mathcal{L}}{\delta(\partial_\kappa \phi^\ell)} \frac{i}{2} \omega^{\mu\nu} (\mathcal{J}_{\mu\nu})^\ell{}_m \partial_\kappa \phi^m + \frac{\delta \mathcal{L}}{\delta(\partial_\kappa \phi^\ell)} \omega_\kappa{}^\lambda \partial_\lambda \phi^\ell$$

Setting the coefficient of $\omega^{\mu\nu}$ equal to 0

$$\rightarrow \quad 0 = \frac{i}{2} \frac{\delta \mathcal{L}}{\delta \phi^\ell} (\mathcal{J}_{\mu\nu})^\ell{}_m \phi^m$$
$$+ \frac{i}{2} \frac{\delta \mathcal{L}}{\delta(\partial_\kappa \phi^\ell)} (\mathcal{J}_{\mu\nu})^\ell{}_m \partial_\kappa \phi^m + \frac{1}{2} \frac{\delta \mathcal{L}}{\delta(\partial_\kappa \phi^\ell)} (\eta_{\kappa\mu} \partial_\nu - \eta_{\kappa\nu} \partial_\mu) \phi^\ell \tag{4}$$

Using Euler-Lagrange eqs.

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi^\ell)} = \frac{\delta \mathcal{L}}{\delta \phi^\ell}$$

and eq.

$$T^\nu{}_\mu = \frac{\delta \mathcal{L}}{\delta(\partial_\nu \phi^\ell)} \partial_\mu \phi^\ell - \delta^\nu_\mu \mathcal{L}$$
$$\rightarrow T_{\nu\mu} = \eta_{\nu\sigma} \frac{\delta \mathcal{L}}{\delta(\partial_\sigma \phi^\ell)} \partial_\mu \phi^\ell - \eta_{\nu\mu} \mathcal{L}$$

for energy-momentum tensor, gives

$$(4) \longrightarrow 0 = \partial_\kappa \left[\frac{i}{2} \frac{\partial \mathcal{L}}{\delta(\partial_\kappa \phi^\ell)} (\mathcal{J}_{\mu\nu})^\ell{}_m \phi^m \right] + \frac{1}{2} (T_{\mu\nu} - T_{\nu\mu}) \tag{5}$$

This suggests to define new energy-momomentum tensor:

$$\Theta^{\mu\nu} = T^{\mu\nu} + \frac{i}{2}\partial_k \left[\frac{\delta \mathcal{L}}{\delta(\partial_k \phi^e)} (\mathcal{G}^{\mu\nu})^e{}_m \phi^m - \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi^e)} (\mathcal{G}^{k\nu})^e{}_m \phi^m - \frac{\delta \mathcal{L}}{\delta(\partial_\nu \phi^e)} (\mathcal{G}^{k\mu})^e{}_m \phi^m \right]$$

$\rightarrow [\dots]^{k\mu\nu}$ is anti-sym in μ, k

$$\rightarrow \partial_\mu \Theta^{\mu\nu} = \underbrace{\partial_\mu T^{\mu\nu}}_{=0} + \frac{i}{2} \underbrace{\partial_k \partial_\mu [\dots]^{k\mu\nu}}_{=0}$$

$= 0$ "conserved"

also when $\mu = 0$ ($\rightarrow k \in \{1,2,3\}$), then

$$\int \Theta^{0\nu} d^3x = \int T^{0\nu} d^3x = P^\nu$$

where $P^0 = H$

$\rightarrow \Theta^{\mu\nu}$ can be regarded as "energy-momentum" tensor

But this tensor is – contrary to $T^{\mu\nu}$ – <u>symmetric</u> in μ and ν:

$$\Theta^{\mu\nu} - \Theta^{\nu\mu}$$

$$= T^{\mu\nu} - T^{\nu\mu} + i\partial_k \left[\frac{\delta \mathcal{L}}{\delta(\partial_k \phi^e)} (\mathcal{G}^{\mu\nu})^e{}_m \phi^m \right] \overset{(5)}{=} 0$$

<u>Note:</u> It is $\Theta^{\mu\nu}$ (rather than $T^{\mu\nu}$) that acts as a source of the gravitational field!

Since $\Theta^{\mu\nu}$ is symmetric, we can define

$$\mathcal{M}^{\lambda\mu\nu} := x^\mu \Theta^{\lambda\nu} - x^\nu \Theta^{\lambda\mu}$$

$$\longrightarrow \quad \partial_\lambda \mathcal{M}^{\lambda\mu\nu} = \Theta^{\mu\nu} - \Theta^{\nu\mu} \overset{(5)}{=} 0$$

$$\longrightarrow \quad J^{\mu\nu} = \int \mathcal{M}^{0\mu\nu} d^3x = \int d^3x \left(x^\mu \Theta^{0\nu} - x^\nu \Theta^{0\mu}\right)$$

is conserved!

We have for $J_k := \frac{1}{2}\varepsilon_{ijk} J^{ij}$:

- $[H, \vec{J}] = 0$
- $$[P_j, J_i] = \frac{1}{2}\varepsilon_{iek}[P_j, J^{ek}]$$
  $$= -\frac{i}{2}\varepsilon_{iek}\int d^3x \left(x^e \frac{\partial}{\partial x^j}\Theta^{0k} - x^k \frac{\partial}{\partial x^j}\Theta^{0e}\right)$$
  $$= +i\varepsilon_{ijk}\int d^3x\, \Theta^{0k} = -i\varepsilon_{ijk}P_k$$
- Set $K_k := J^{k0} \longrightarrow K_k = \int d^3x \left(x^k \Theta^{00} - x^0 \Theta^{0k}\right)$

  or $\vec{K} = -t\vec{P} + \int d^3\vec{x}\, \Theta^{00}(\vec{x}, t)$

  conservation of $J^{\mu\nu} \rightarrow \dot{\vec{K}} = 0 = -\vec{P} + i[H, \vec{K}]$

  thus $[H, \vec{K}] = -i\vec{P}$
- also can show $[P_j, K_k] = -i\,\delta_{jk} H$

§3.3 Gauge invariance

Recall from §1.4 that the massive Maxwell field $A_\mu(x)$ is given as operator :

polarization

$$A_\mu(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \sum_{\sigma=1}^{3} \left[e^{ik\cdot x} \varepsilon_\mu^{(\sigma)}(k) a(k) + e^{-ik\cdot x} \varepsilon_\mu^{(\sigma)*} a^\dagger(k) \right]$$

We had also seen that the corresp. propagator is given by :

$$D_{\mu\nu}(x,y) = \langle 0 | T[A_\mu(x) A_\nu(y)] | 0 \rangle$$
$$= \int \frac{d^4q}{(2\pi)^4} e^{iq\cdot(x-y)} \frac{-\eta_{\mu\nu} + q_\mu q_\nu/m^2}{q^2 - m^2 + i\varepsilon}$$

$\longrightarrow$ singularity for $m=0$!

So $D_{\mu\nu}$ is Lorentz covariant but ill-defined for $m=0$

Deeper reason: there is no Lorentz covariant 4-vector A_μ which is massless !

Instead one can show:

$$U(\Lambda) A_\mu(x) U^{-1}(\Lambda) = \Lambda_\mu{}^\nu A_\nu(\Lambda x) + \underbrace{\partial_\mu \Omega(x,\Lambda)}_{\text{gauge trf.}}$$

→ require that the part of action S for matter and its interaction with radiation be invariant under

$$A_\mu(x) \longrightarrow A_\mu(x) + \partial_\mu \varepsilon(x)$$

This implies a change of action:

$$\delta S = \int d^4x \, \frac{\delta S}{\delta A_\mu(x)} \partial_\mu \varepsilon(x)$$

→ Lorentz invariance requires

$$\partial_\mu \frac{\delta S}{\delta A_\mu(x)} = 0 \qquad (*)$$

(*) is true if S only involves

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$$

then $$\frac{\delta S}{\delta A_\mu(x)} = 2\partial_\nu \frac{\delta S}{\delta F_{\mu\nu}(x)}$$

$$\left(\longrightarrow \partial_\mu \frac{\delta S}{\delta A_\mu(x)} = 2\partial_\mu \partial_\nu \frac{\delta S}{\delta F_{\mu\nu}(x)} = 0 \text{ as } F_{\mu\nu} \text{ anti-sym.} \right)$$

But if S involves A_μ itself, (*) is a non-trivial constraint!

<u>Solution</u>: We have seen that infinitesimal internal symmetries of action S imply existence of conserved currents!

In particular, when

$$\delta\phi^\ell(x) = i\varepsilon(x) q_\ell \phi^\ell(x) \qquad (**)$$

leaves S invariant for constant ε, then

$$\delta S = -\int d^4x\, J^\mu(x)\partial_\mu \varepsilon(x)$$

and when matter fields satisfy their equations of motion, then

$$\partial_\mu J^\mu = 0$$

We saw last time that in case $(**)$, we have

$$J^\mu = -i \sum_\ell \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi^\ell)} q_\ell \phi^\ell$$

and $\quad [Q, \phi^\ell(x)] = -q_\ell \phi^\ell(x)$

where $\quad Q = \int d^3x\, J^0$

$\longrightarrow$ set $\quad \dfrac{\delta S}{\delta A_\mu(x)} = J^\mu(x) \quad$ in $(*)$

then under gauge trfs. :

$$\delta S = \int d^4x\; \partial_\mu \varepsilon(x) J^\mu(x) = 0$$

$\longrightarrow$ matter action is invariant under joint trfs.:

$$\delta A_\mu(x) = \partial_\mu \varepsilon(x), \qquad \delta \phi_\ell(x) = i\varepsilon(x) q_\ell \phi_\ell(x) \quad (1)$$

A symmetry of type (1) is called a "local symmetry", whereas for constant ε the symmetry is called "global".

Action for photons themselves:

$$S_\gamma = -\frac{1}{4}\int d^4x\, F_{\mu\nu} F^{\mu\nu}$$

(unique gauge-invariant functional that is quadratic in $F_{\mu\nu}$)

$\longrightarrow$ field equation for electromagnetism now reads:

$$0 = \frac{\delta}{\delta A_\nu}\left[S_\gamma + S_M\right] = \partial_\mu F^{\mu\nu} + j^\nu$$

homogeneous Maxwell eqs.:

$$0 = \partial_\mu F_{\nu\varepsilon} + \partial_\varepsilon F_{\mu\nu} + \partial_\nu F_{\varepsilon\mu}$$

(follow directly from definition $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$)

<u>Notation:</u> $\quad D_\mu \phi^\ell := \partial_\mu \phi^\ell - i q_{f_\ell} A_\mu \phi^\ell$

$$\longrightarrow \delta D_\mu \phi^\ell = \delta \partial_\mu \phi^\ell - i q_{f_\ell}\, \delta A_\mu\, \phi^\ell - i q_{f_\ell} A_\mu \delta\phi^\ell$$

$$= i\varepsilon q_{f_\ell} \partial_\mu \phi^\ell \cancel{+ i q_{f_\ell} \phi^\ell \partial_\mu \varepsilon - i q_{f_\ell} \partial_\mu \varepsilon\, \phi^\ell} - i q_{f_\ell} A_\mu i\varepsilon q_{f_\ell} \phi^\ell$$

$$= i\varepsilon(x)\, q_{f_\ell} D_\mu \phi^\ell(x)$$